Modulation Graphs in Popular Music

Jason I. Brown and Ian C. George

Graph theory offers a unique perspective into the sequential nature of music. For example, Alisa Crans et al. [2] explore the actions of the dihedral group on the set of major and minor triads, which are elegantly represented on the triangular lattice known as the Neo-Riemannian Tonnetz [13], on Douthett and Steinbach's hexagonal lattice [3], and on Waller's torus [15]. Graphs also serve as a way to study voice leading [19].

In addition, graphs can be used to model the possibilities of tonal modulation; it is this last idea that is of special interest to us. Tonal modulation is the act of changing tonalities in a composition, which can be central to the emotional portrait of a piece of music. Examples of modulation can be found in "Here, There and Everywhere" by the Beatles (G major → B♭ major → G major) and "Money, Money, Money" by ABBA (A minor → B♭ minor). Graphs can be constructed with the vertices representing keys and the (directed) edges connecting pairs of vertices for which modulations either are possible or are present in a database of songs.

Adrian Walton [16] introduced two such graphs, one based on major scales, the other on both major and natural minor scales, and restricted modulation to what is "simple" in the sense that one changes key via a shared (or "pivot") major or minor chord. We consider here a broader range of pivot modulation graphs, based on an extended set of scales (and modes). We consider the combinatorial properties of such graphs and perform an analysis of modulations present in the discography of the Beatles to bridge our theoretical model to what is done in practice.

Ultimately, it is a composer's discretion as to what modulations take place and how in a composition. Our aim is to set modulations in a mathematical setting, with the expectation that a combinatorial view might provide new musical insights and options.

Musical and Mathematical Background

We assume that the reader is acquainted with the basics of Western music theory and graph theory (see, for example, [17] for the latter). We introduce the following numeric labeling of the octave: $\mathrm{C} \leftrightarrow 0$, $\mathrm{D}\flat = \mathrm{C}\sharp \leftrightarrow 1, \ldots, \mathrm{B} \leftrightarrow 11$, a bijection between the pitch classes (which we often refer to as notes) and $\mathbb{Z}_{12}$; addition of pitch classes corresponds to addition modulo 12.

To create an interesting musical landscape, a sequence of notes can be chosen to form a scale whose type is invariant under addition modulo 12. Taking all twelve notes of the octave leads to the chromatic scale, but the most commonly used scales, such as the family of diatonic scales (i.e., major, minor, and their modes), contain seven notes. The initial note of a scale is called the tonic, providing a sense of stability or "home" for the scale. The note of the scale that forms the essential foundation of a song (or part of a song) is called the key (while there are algorithms to determine the key of a song, an accurate account of the key of a song often requires musical analysis of the melody and chords).

There are five scales of primary importance in pop and rock music that are worth mentioning for later discussion of modulation.

- The first, and perhaps the most fundamental to Western music, is the major scale: if i is taken to be the tonic, then the major scale is formed as

  $$i, i+2, i+4, i+5, i+7, i+9, i+11\,,$$

  where addition, as throughout this paper, is taken modulo 12. For example, the C major scale has root C and contains the notes C, D, E, F, G, A, B.
- Next, we have the Mixolydian scale: starting from the tonic i, the scale contains

  $$i, i+2, i+4, i+5, i+7, i+9, i+10\,;$$

  for example, the C Mixolydian scale is C, D, E, F, G, A, B♭. The Mixolydian scale differs from the major scale only by a flattened seventh note. One can view this scale as consisting of the notes of the major scale with tonic seven semitones down (in this example F) but starting on its fifth note (i.e., C). Viewed in this way, the Mixolydian scale is often called a mode of the corresponding major scale. The Mixolydian scale plays a prominent role in pop and rock music (e.g., the verses of "Norwegian Wood" and the ending of "Hey Jude," by the Beatles, "You Really Got Me," by the Kinks, and "Takin' Care of Business," by BTO).
- Another scale that is a mode of the major scale is the natural minor scale. This scale is built on the sixth note of a major scale and is also known as the Aeolian mode. The natural minor scale with tonic i consists of the notes

  $$i, i+2, i+3, i+5, i+7, i+8, i+10\,.$$

https://doi.org/10.1007/s00283-023-10325-y

The C natural minor scale is C, D, E♭, F, G, A♭, B♭. The relative minor scale of a major scale with tonic i has tonic $i-3$ and shares exactly the same notes as its relative major scale.

- A variation of the natural minor scale is the harmonic minor scale, with a sharpened seventh note: starting on the tonic i, the scale consists of the notes

$$i, i+2, i+3, i+5, i+7, i+8, i+11\,.$$

Hence, the C harmonic minor scale is C, D, E♭, F, G, A♭, B. The harmonic minor scale is prominent in music originating in Eastern Europe, but it has permeated popular music as well (e.g., "Girl," by the Beatles, "California Dreamin'," by the Mamas and the Papas, and "Sultans of Swing," by Dire Straits).

- Lastly, we mention another variation of the natural minor, called the melodic minor. This scale has sharpened sixth and seventh notes, so this scale can be thought of alternatively as a major scale with its third note flattened. The form of the melodic minor is

$$i, i+2, i+3, i+5, i+7, i+9, i+11\,,$$

with the C melodic minor being C, D, E♭, F, G, A, B. (In some settings, the melodic minor in descending form contains the notes of the natural minor scale with the same tonic). Melodic minor scales appear often in jazz settings, but they may appear in pop music as well (e.g., "Come Together" and "While My Guitar Gently Weeps," by the Beatles).

Scales provide a way to build chords with three notes (triads). Suppose our scale is the sequence $j_0, j_1, \dots, j_6$. Then for $k \in \{0, 1, \dots, 6\}$, the set $\{j_k, j_{k+2}, j_{k+4}\}$ (addition modulo 7 here) is a (diatonic) triad (or simply a chord in our context) of the scale. In this way, every note of a scale has a corresponding diatonic triad built on it, and a note of the scale appears in three such triads. Depending on the context, a diatonic triad may be major (i.e., of the form $\{i, i+4, i+7\}$), minor (of the form $\{i, i+3, i+7\}$), diminished (of the form $\{i, i+3, i+6\}$), or augmented (of the form $\{i, i+4, i+8\}$). (It is also important to consider transposition, which is the application of a function $f_k : x \overset{x}{\longrightarrow} +k$, where addition is of course modulo 12, for some fixed $k \in \mathbb{Z}_{12}$. Transposition preserves the nature of both scales and chords.)

What does it mean to modulate in a piece of music? A modulation is a change of key at some moment in a composition. According to the *Berklee Book of Jazz Harmony* [10], there are a number of ways to achieve a modulation. Direct modulation involves reaching a new key abruptly, with no transition. In a pivot modulation, a chord common to an original and target key is used in a progression to reach the target. A third kind of modulation, called transitional, involves repeating harmonic progressions (i.e., sequences) that bridge two keys in a keyless fashion.

Our focus in this paper will be on pivot modulation. For some mathematical clarity, let us introduce the following notation: for a scale S, let $T(S)$ denote the set of diatonic triads from S.

Definition 1. Given two scales S and P, we say that there exists a (pivot) modulation between S and P (or between P and S) if $T(S) \cap T(P) \neq \emptyset$. If a modulation exists, we call every $p \in T(S) \cap T(P)$ a pivot chord (or pivot triad) of S and P.

This treatment of modulation is very similar to that from [16], except here we permit diminished and augmented triads to be pivot chords. While these types of diatonic chords do not play a role when one restricts to pivots between major scales or natural minor scales (since each such scale contains no augmented chord and a unique diminished chord that is shared with no other except the relative major/minor), such chords will play a role in our discussion of our extended scale collection.

Pivot Modulation Graphs

From our definition of the existence of a pivot modulation above, we can now give a construction of a simple graph of a pivot modulation: a collection of scales Σ is taken to be the vertex set, with an edge $e = \{S_1, S_2\}$ if and only if S_1 and S_2 are distinct scales in Σ with $T(S_1) \cap T(S_2) \neq \emptyset$, that is, if and only if S_1 and S_2 share a (pivot) chord. If E is the set of all such edges, then the pivot graph $G = (\Sigma, E)$ is a simple graph, that is, without multiple edges.

Alternatively, we can form a multigraph in a similar fashion, where we allow $|T(S_1) \cap T(S_2)|$ edges between vertices S_1 and S_2. Each of these edges can be labeled with a unique element from $T(S_1) \cap T(S_2)$. Such a multigraph can be useful in examining sequences of pivot modulations as walks in the corresponding multigraph. We may also wish to "combine" several scales so that they are represented by a single vertex. To do so, we simply take the union of triad sets for each scale: $T(F) = \bigcup_{S \in F} T(S)$ for a collection of scales F.

Now that we have shown how to form a graph of pivot modulation, how might such graphs be of use or interest? First and foremost, they are a novel tool for visualizing modulation. We may want to know some combinatorial properties of these graphs such as diameter and clique number, or more physical properties including cliques, independent sets, and walks. From a purely mathematical point of view, such information is intriguing, but it can also be useful for a musician or composer. For example, suppose that a composer wants to pass from one key to another, passing through two intermediate keys. A sequence of pivot modulations with the desired effect may be found by examining the walks of length three between the initial and target keys of a pivot modulation graph. If a composer wishes to select a number of keys between any two of which there is a modulation, the cliques of a graph would give such groupings. In this section we aim to discuss graphs that represent a realistic framework for pivot modulation in pop music and some of their properties. We remark that our approach here extends that of [16], where only pivot modulation graphs related to major scales and

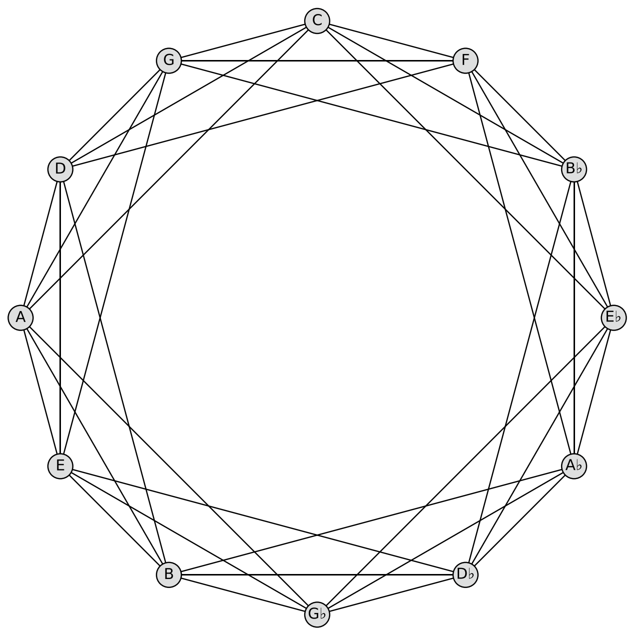


Figure 1. The 12-vertex pivot modulation graph, where each vertex represents the major and Mixolydian scales sharing a tonic. This graph can be described as the circulant graph $C_{12}(2, 3, 5)$ with distances measured in semitones.

natural minor scales are considered (we do not revisit those here). Our extension was motivated by the fact that some of the best pop songs involve other scales and that mathematical tools for modulation in those settings would be very useful.

Major/Mixolydian Scales

The first pivot modulation graph we would like to consider is the simple graph formed from major scales and their Mixolydian modes (see Figure 1). In this graph, a given vertex represents a major scale and the corresponding Mixolydian mode (i.e., the one sharing the same tonic). For example, the C vertex represents the C major scale and the C Mixolydian scale. We may notate such a pairing with a capital M, for major, subscripted with the numeric label of the relevant tonic: M_i. Our decision to combine these two scales into a single vertex comes from their compatibility: as noted earlier, the major and Mixolydian scales differ by only a single note. In several genres of modern music, such as rock 'n' roll and pop, borrowing chords from each scale to create a hybrid musical landscape of major/Mixolydian is especially common. Thus we have twelve vertices in our graph, which we will denote by their tonics.

This pivot modulation graph is in fact the circulant graph $C_{12}(2, 3, 5)$ (a circulant graph $C(n, S)$ has vertex set $\mathbb{Z}_n$ with $S \subseteq \mathbb{Z}_n$ and edge set $\{\{i, i+s\} : s \in S\}$). Observe that the arrangement of vertices follows the circle of fifths.

This graph has diameter 2, independence number 3, and clique number 4. Thus to modulate from a given major/Mixolydian scale to any other, at most one intermediate modulation is required, the largest number of scales of which any pair does not have a modulation is three, and there are at most four scales such that all pairs have a modulation between them.

Table 1. Maximal independent sets of the 12-vertex major/Mixolydian pivot modulation graph.

Class of Set	Count
$\{M_i, M_{i+1}\}$	12
$\{M_i, M_{i+6}\}$	6
$\{M_i, M_{i+4}, M_{i+8}\}$	4

Table 2. Maximal cliques of the 12-vertex major/Mixolydian pivot modulation graph.

Class of Set	Count
$\{M_i, M_{i+2}, M_{i+7}, M_{i+9}\}$	12

Let us now examine the maximal independent sets and cliques of this graph. There are two classes of maximal independent set with two vertices: the first contains vertices a semitone apart, and the other vertices a tritone apart. The only maximal independent sets (that is, those of maximum cardinality) are formed from scales whose tonics form augmented triads, of which there are four (Table 1). All of the maximal cliques are also maximal (and of cardinality 4) and of one class, having the form of a sequence of four scales whose tonics are consecutively separated by a perfect fifth (Table 2).

As for the automorphisms of the graph, they are described by the following group generators: $(M_0, M_1, M_2, \ldots, M_{11}), (M_1, M_{11})(M_2, M_{10})(M_3, M_9)(M_4, M_8)(M_5, M_7)$. This group is isomorphic to D_{12}, the dihedral group on twelve elements.

Minor Scales

Now we turn our attention to minor scales, in their natural, harmonic, and melodic forms. The graph we consider in this section is again one with twelve vertices (see Figure 2). Now, however, its vertices will be the union of diatonic triad sets from the natural, harmonic, and melodic minor scales sharing a tonic. Our justification for such a choice is in the same spirit as before: these flavors of minor scales may be easily interwoven. According to [10], "Minor key jazz tunes are rarely diatonic to just one of these scales. Typically, chords from several of these sources are used interchangeably in minor key compositions." Examples in the Beatles' oeuvre of songs based on more than one minor scale include "You Never Give Me Your Money" (natural and harmonic minors) and "Because" (natural, harmonic, and melodic minors). Thus to keep to a framework that is realistic for pivot modulation, we will adopt the practice of grouping each minor scale into a single vertex. This grouping of scales, similar to what we have done above, can be denoted by m_i, where i is the tonic of each minor scale.

This pivot modulation graph can be simply described as the circulant graph $C_{12}(1, 2, 3, 4, 5)$, that is, the complete graph K_{12} minus a perfect matching

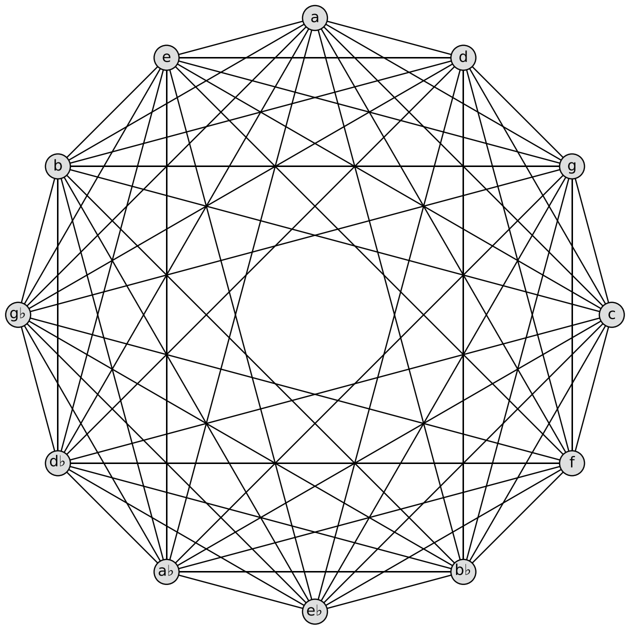


Figure 2. The 12-vertex pivot modulation graph, where each vertex represents the natural, harmonic, and melodic minor scales sharing a tonic. This graph can be described as the circulant graph $C_{12}(1, 2, 3, 4, 5)$ or the complete graph K_{12} without a perfect matching.

$\{\{m_0, m_6\}, \{m_1, m_7\}, \{m_2, m_8\}, \{m_3, m_9\}, \{m_4, m_{10}\}, \{m_5, m_{11}\}\}$, the set of edges joining vertices separated by a tritone (that is, by six semitones). Thus one can modulate, in a single step, from a minor scale grouping to any other except the scale a tritone away. Hence it is clear that this graph has diameter 2, which is equal to its independence number. In contrast, it has clique number 6.

This highly connected graph permits only a single class of maximal independent set, that of the tritone-paired vertices (Table 3). However, there are a number of classes of maximal cliques. Each of the six classes of maximal cliques are of maximum size, with two notable classes being vertices forming a semitone sequence (i.e., six consecutive notes separated by a semitone), and vertices separated by intervals of a perfect fifth in sequence (Table 4).

We describe the group of automorphisms in terms of its generators:

$$(m_0, m_1)(m_6, m_7), (m_1, m_2)(m_7, m_8), (m_2, m_3)(m_8, m_9),$$
$$(m_3, m_4)(m_9, m_{10}), (m_4, m_5)(m_{10}, m_{11}), (m_5, m_{11}).$$

This automorphism group has order 46,080, with 2 the order of each generator.

Both Types of Scales

Thus far, we have considered only graphs of pivot modulation involving strictly major or minor flavors of scales. To fail to consider a case in which modulation between flavors is possible would be to ignore much of the creative realm when it comes to modulation. Therefore, in this section we examine the 24-vertex graph containing vertices of both major and minor type used previously. To distinguish between the two types of vertices in the graph, as before, we will use uppercase letters to denote tonics of major scale vertices and lowercase letters for the tonics of minor scale vertices (see Figure 3).

The outside ring of the graph contains exactly the vertices of major type, and thus the subgraph induced by the outer ring is the pivot modulation graph explored above. Similarly, the inside ring contains exactly the vertices of minor type, and thus its induced subgraph is the graph explored in the previous subsection. It is the edges between these rings that give a new and nontrivial pivot modulation graph. As has been the case thus far, this graph has diameter 2. It shares with the 12-vertex major pivot modulation graph independence number 3, while uniquely bearing clique number 10.

This graph has seven classes of maximal independent sets, with all but one containing only two vertices. We are now able to see independent sets containing vertices of both major and minor type. Pairs of vertices separated by a tritone appear as maximal independent sets, both major, both minor, or one of each. A pair of vertices of major type separated by a semitone forms a maximal independent set, as does a vertex of each type with the minor vertex a single semitone or eight semitones higher. The only class of maximal independent set containing three vertices is composed of vertices of major type that form an augmented triad (Table 5).

There are six classes of maximal cliques in this graph, two of which contain only vertices of minor type. Also of interest is the class of maximal cliques with four vertices in each set that are of major type, separated by consecutive perfect fifths. The remaining six vertices are of minor type, and similarly are separated by consecutive perfect fifths. Note that splitting these sets into strictly their major or minor vertices forms a class of maximal sets in their appropriate 12-vertex graph (Table 6).

The automorphism group is isomorphic to D_{12}, with the explicit generators

$$(M_i, M_{i+1}, \dots, M_{i+11})(m_i, m_{i+1}, \dots, m_{i+11})$$

Table 3. Maximal independent sets of the 12-vertex minor scale pivot modulation graph.

Class of Set	Count
$\{m_i, m_{i+6}\}$	6

Table 4. Maximal cliques of the 12-vertex minor scale pivot modulation graph.

Class of Set	Count
$\{m_i, m_{i+1}, m_{i+2}, m_{i+3}, m_{i+4}, m_{i+5}\}$	12
$\{m_i, m_{i+1}, m_{i+2}, m_{i+4}, m_{i+5}, m_{i+9}\}$	12
$\{m_i, m_{i+1}, m_{i+2}, m_{i+3}, m_{i+5}, m_{i+10}\}$	12
$\{m_i, m_{i+1}, m_{i+2}, m_{i+5}, m_{i+9}, m_{i+10}\}$	12
$\{m_i, m_{i+2}, m_{i+4}, m_{i+7}, m_{i+9}, m_{i+11}\}$	12
$\{m_i, m_{i+3}, m_{i+4}, m_{i+7}, m_{i+8}, m_{i+11}\}$	4

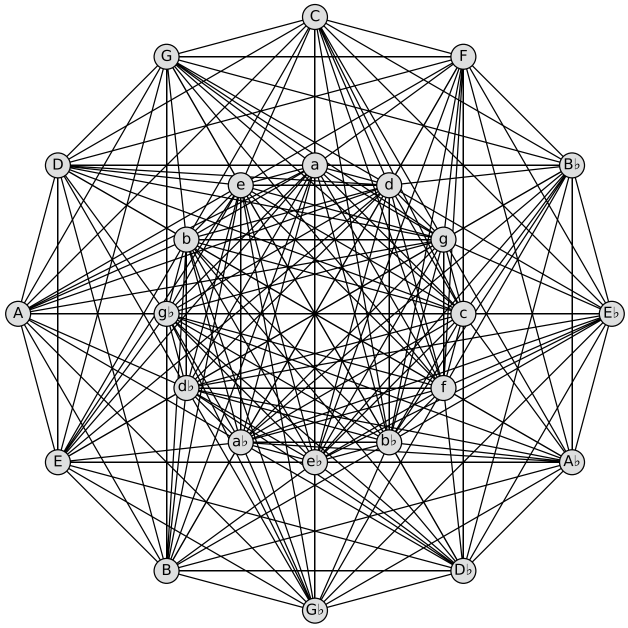


Figure 3. The 24-vertex pivot modulation graph containing the vertices of both major (outer ring) and minor type (inner ring).

and

$$(M_1, M_{11})(M_2, M_{10})(M_3, M_9) \cdots (M_4, M_8)(M_5, M_7)\cdot$$
$$(m_0, m_2)(m_3, m_{11})(m_4, m_{10})(m_5, m_9)(m_6, m_8)\,.$$

Table 5. Maximal independent sets of the 24-vertex pivot modulation graph combining vertices of major and minor type.

Class of Set	Count
$\{M_i, M_{i+1}\}$	12
$\{M_i, m_{i+1}\}$	12
$\{M_i, m_{i+6}\}$	12
$\{M_i, m_{i+8}\}$	12
$\{m_i, m_{i+6}\}$	6
$\{M_i, M_{i+4}, M_{i+8}\}$	4

Table 6. Maximal cliques of the 24-vertex pivot modulation graph combining vertices of major and minor type.

Class of Set	Count
$\{m_i, m_{i+1}, m_{i+2}, m_{i+3}, m_{i+4}, m_{i+5}\}$	12
$\{m_i, m_{i+1}, m_{i+4}, m_{i+5}, m_{i+8}, m_{i+9}\}$	4
$\{M_i, m_i, m_{i+2}, m_{i+3}, m_{i+4}, m_{i+7}, m_{i+11}\}$	12
$\{M_i, m_i, m_{i+2}, m_{i+3}, m_{i+7}, m_{i+10}, m_{i+11}\}$	12
$\{M_i, M_{i+5}, m_i, m_{i+2}, m_{i+3}, m_{i+4}, m_{i+5}, m_{i+7}\}$	12
$\{M_i, M_{i+2}, M_{i+7}, M_{i+9}, m_i, m_{i+2}, m_{i+4}, m_{i+7}, m_{i+9}, m_{i+11}\}$	12

Modulation in the Beatles' Music

The Beatles hold an impressive discography across a decade of active songwriting. A body of work has formed for studying their music or using it in catalogues for data research (for example, [1, 7, 12, 18]). In this section we will use the music of the Beatles to connect our theoretical pivot modulation graphs to what is done in practice.

We begin our study of modulation in the Beatles' music with an analysis of 183 songs across their discography. This list contains the songs from the isophonics musical annotation project[1] and other singles. Each song was examined for modulations of key according to the framework outlined above. This involved following changes in chords for each song, obtained through the isophonics project or with cross-reference between *Ultimate Guitar Tabs* [14] and *The Beatles: Complete Scores* [5]. The key(s) present in a song were determined manually through musical analysis.

All modes of modulation (direct, transitional, and pivot) were considered, since we wished to find what was actually done in practice.

Out of our list of 183 songs, 77 were found to have at least one modulation. If a modulation appeared more than once in a song, it was recorded only once; these modulations would not be unique songwriting decisions, but rather a single decision that is repeated to fit the conventions of song structure. Once the modulations of a song were found, pivot chords for each modulation were considered. If more than a single chord could be a pivot, then the one that felt most like the pivot was chosen. If there was no possibility for a pivot chord, i.e., the modulation was direct or transitional, then it was noted that no pivot chord was used. Chords that extend beyond triads (e.g., seventh and ninth chords) were treated as the fundamental triad of their structure. For example, the modulations in "Think for Yourself," by George Harrison, can be expressed by $\mathrm{A}_{\mathrm{minor}} \rightarrow \mathrm{G}_{\mathrm{major}} \rightarrow \mathrm{A}_{\mathrm{minor}}$; the verses of this song are in the key of A minor, while the choruses are in G major. Thus we have modulation from A minor to G major and from G major to A minor. In the former, a pivot triad of C major is used, while in the latter the pivot is A minor.

All of the modulations found in the Beatles' songs can be seen in the graph in Figure 4. We will use the vertex arrangement from the full pivot modulation graph in Figure 2 to show what modulations they use. Since there was no a priori guarantee that all the modulations observed were reversed, the graph formed is directed. Edges that are blue represent those permitted by pivot modulation, and red represents the remainder (possible only through direct modulation). A dashed edge indicates a modulation that occurred only through use of a pivot, a dotted edge indicates a modulation that occurred only directly and without a pivot, while a dotted and dashed edge marks a modulation that occurred both with and without a pivot. There are 89 directed edges in the graph, with 59 strictly pivot modulations, 18 strictly direct modulations, and 12 of both mechanisms.

[1]See http://isophonics.net/content/reference-annotations-beatles.

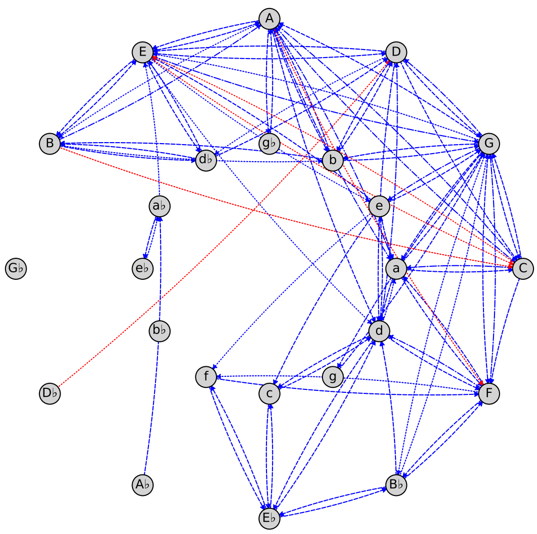


Figure 4. Modulations found in the music of the Beatles. Blue edges indicate modulations allowed by a pivot, while red indicates the remainder. Some modulations were performed only with pivots (dashed), some only by direct modulation (dotted), and in both ways (dotted and dashed).

Only four modulations across three songs are not permitted by a pivot; hence nonpivoting modulations are rare occurrences in the Beatles' canon. Interestingly, there are two vertices not involved in any modulation: G♭ major and B♭ minor. In contrast, the most active vertex is G, having in-degree 9 and out-degree 10.

We are also interested in modulations of a certain class. Intuitively, modulations such as $\mathrm{A}_{\mathrm{minor}} \rightarrow \mathrm{C}_{\mathrm{major}}$ and $\mathrm{E}_{\mathrm{minor}} \rightarrow \mathrm{G}_{\mathrm{major}}$ should be of the same class, because both initial and destination keys are the same flavor of scale, and the semitone distances between the initial keys equal those of the destination keys. Given a modulation $x_i \rightarrow y_j$, where $x, y \in \{M, m\}$ and $i, j \in \mathbb{Z}_{12}$, the set of transposed modulations $x_{i+k} \rightarrow y_{j+k}$ $(k \in \mathbb{Z}_{12})$ forms its equivalence class.

The example modulations above belong to the $m_i \rightarrow M_{i+3}$ class. Across the 77 songs found to have a modulation, there are 27 unique classes of modulation (a testament to the harmonic creativity of John, Paul, and George). Three of these classes do not have pivot chords: $M_i \rightarrow M_{i+1}$, $M_i \rightarrow M_{i+4}$, $M_i \rightarrow M_{i-4}$. We also make the following observations: seven classes are from a major vertex to a minor vertex, five are from a minor vertex to a major vertex, nine are between major vertices, and six are between minor vertices. To see the Beatles' modulation tendencies, we counted the number of modulations present in each class from the 77 songs containing modulation. The largest modulation classes are $m_i \rightarrow M_{i+3}$ and $M_i \rightarrow m_{i-3}$, having 27 and 26 instances of modulation, respectively. These classes are simply the common modulations between relative major and minor scales in each direction. The second most counted modulation classes are $M_i \rightarrow m_i$, $m_i \rightarrow M_i$ (parallel pairs), $M_i \rightarrow M_{i+3}$, $M_i \rightarrow M_{i-3}$, and $M_i \rightarrow M_{i+5}$, all with 13 instances. The reverse modulation of the final class, $M_i \rightarrow M_{i+7}$, has a similar count of 11. Figure 5 summarizes the different classes of modulations in the Beatles' songs.

Considering classes of modulation can allow us to identify structures of modulation present in the Beatles' music. Using modulation between vertices of the form M_i or m_i, we can form a small directed graph that entirely describes the modulations of a song. Let us illustrate this with an example. The song "Doctor Robert" contains modulations $\mathrm{A}_{\mathrm{major}} \rightarrow \mathrm{B}_{\mathrm{major}}$ and $\mathrm{B}_{\mathrm{major}} \rightarrow \mathrm{A}_{\mathrm{major}}$. More generally,

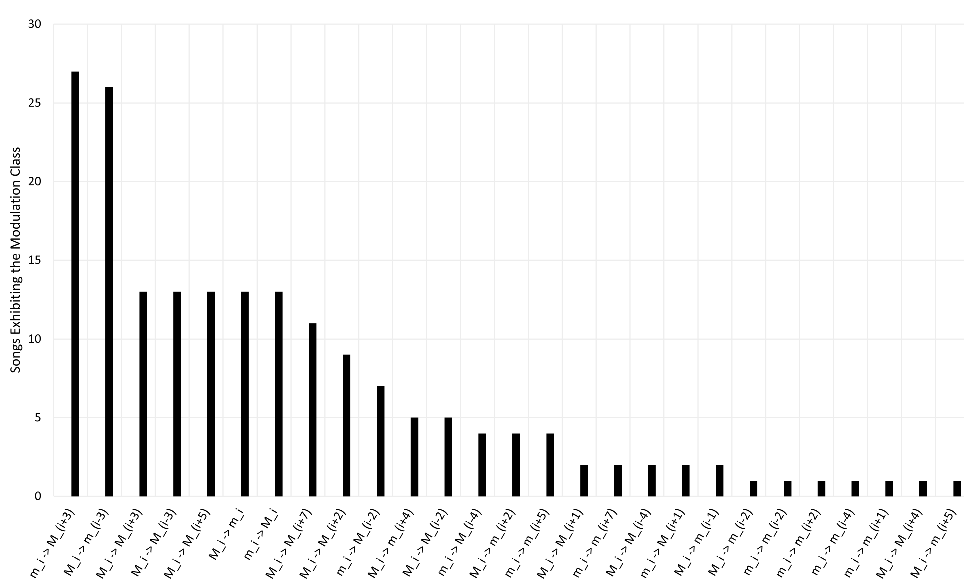
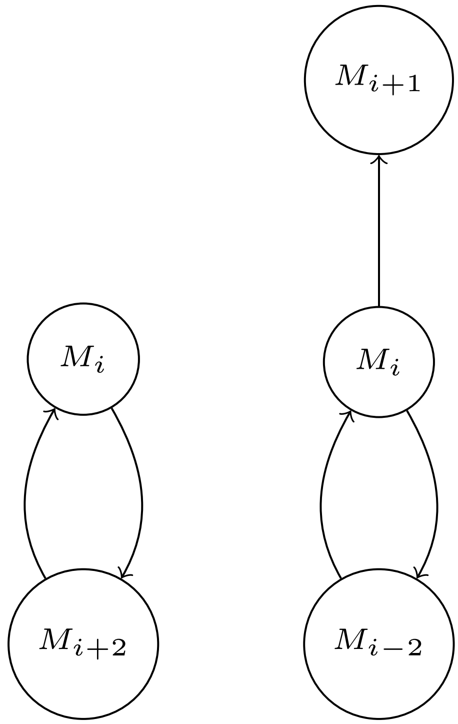


Figure 5. Counting the number of song appearances for each modulation class present in the Beatles' music.



(a) Graph depicting modulation for "Doctor Robert" and "Blackbird."

(b) Graph depicting modulation for "Good Day Sunshine."

Figure 6. Two examples of directed graphs that minimally describe the modulations of Beatles songs.

these modulations are $M_i \rightarrow M_{i+2}$ and $M_{i+2} \rightarrow M_i$. However, "Blackbird" contains modulations $\mathrm{G}_{\mathrm{major}} \rightarrow \mathrm{F}_{\mathrm{major}}$ and $\mathrm{F}_{\mathrm{major}} \rightarrow \mathrm{G}_{\mathrm{major}}$, which are described in the same way. Therefore, the directed cycle of vertices M_i and M_{i+2} describes the modulation activity of both of these songs. The graph that describes the modulations of "Good Day Sunshine" contains the directed cycle between M_i and M_{i+2}, but it is a proper containment, since there is a different type of modulation as well (see Figure 6). There are 35 unique directed graphs across all songs, illustrating the variety of modulations in the songs.

Discussion

The discography of the Beatles provided a sizable catalogue of songs whose modulations could be analyzed through our modulation framework. We were able to capture the majority of modulations present in these songs with our rules of pivot modulation. Noted in the spreadsheet of song analyses is the occasional use of the Neapolitan sixth, or the tritone substitution, as a method for modulation that may not be captured by our framework. This is perhaps an influence of classical music and jazz on the music of the Beatles.

An important factor to our Beatles analysis is human determination of song key and modulations. Our judgments of song key(s) were based on chord transcriptions performed by hand [5, 6, 14] and our own knowledge of music. Of course, such practices may lend themselves to human error. Methods of key or modulation detection by computational and machine-learning techniques [4, 8, 9, 11] may be favored in further graph-theoretic analysis of modulation, especially on very large data sets for the purposes of time efficiency and error reduction.

Future work with graphs of modulation may include examining graphs resulting from different treatments of modulation (such as mentioned above), relating them to other genres of music. Other song catalogues spanning different genres or artists may be of interest, especially to make genre-to-genre comparisons of modulation. We hope that the ideas and results of this paper are of interest to both mathematicians and musicians. Perhaps modulation graphs could be a creative tool for composition and may direct the songwriting choices of composers in ways previously unseen.

Acknowledgments

J. I. Brown acknowledges research support from the Natural Sciences and Engineering Research Council of Canada (NSERC), grant RGPIN 2018-05227.

References

[1] Wei Chai and Barry Vercoe. Music thumbnailing via structural analysis. In *Proceedings of the Eleventh ACM International Conference on Multimedia*, MULTIMEDIA '03, pp. 223–226. ACM, 2003.

[2] Alissa S. Crans, Thomas M. Fiore, and Ramon Satyendra. Musical actions of dihedral groups. *American Mathematical Monthly* 116:6 (2009), 479–495.

[3] Jack Douthett and Peter Steinbach. Parsimonious graphs: a study in parsimony, contextual transformations, and modes of limited transposition. *Journal of Music Theory* 42:2 (1998), 241–263.

[4] Laurent Feisthauer, Louis Bigo, Mathieu Giraud, and Florence Levé. Estimating keys and modulations in musical pieces. In *Sound and Music Computing Conference (SMC 2020)*, Torino, Italy, June 2020, edited by Simone Spagnol and Andrea Valle.

[5] T. Fujita, Y. Hagino, H. Kubo, and G. Sato. *The Beatles: Complete Scores*. Hal Leonard Publishing Corporation, 1993.

[6] Christopher Harte. *Towards Automatic Extraction of Harmony Information from Music Signals*. PhD thesis, Queen Mary University of London, 2010.

[7] Ning Hu, R. B. Dannenberg, and G. Tzanetakis. Polyphonic audio matching and alignment for music retrieval. In *2003 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics (IEEE Cat. No.03TH8684)*, pp. 185–188, 2003.

[8] Filip Korzeniowski and Gerhard Widmer. End-to-end musical key estimation using a convolutional neural network. *CoRR*, arXiv:abs/1706.02921, 2017.

[9] L. Mearns, E. Benetos, and S. Dixon. Automatically detecting key modulations in J. S. Bach chorale recordings. In *8th Sound and Music Computing Conference*, pp. 25–32, 2011.

[10] Joe Mulholland and Tom Hojnacki. *The Berklee Book of Jazz Harmony*. Berklee Press and Hal Leonard Publishing Corp., 2013.

[11] Néstor Nápoles López, Laurent Feisthauer, Florence Levé, and Ichiro Fujinaga. On local keys, modulations, and tonicizations: a dataset and methodology for evaluating changes of key. In *7th International Conference on Digital Libraries for Musicology*, DLfM 2020, pp. 18–26. ACM, 2020.

[12] Drew F. Nobile. Form and voice leading in early Beatles songs. *Music Theory Online* 17:3 (2011).

[13] Hugo Riemann. Ideas for a study "on the imagination of tone." *Journal of Music Theory* 36:1 (1992), 81–117.

[14] Ultimate guitar tabs. Available at https://www.ultimate-guitar.com/.

[15] Derek A. Waller. Combinatorial aspects of the musical chords. *Mathematical Gazette* 62:419 (1978), 12–15.

[16] Adrian Walton. A graph theoretic approach to tonal modulation. *Journal of Mathematics and Music* 4:1 (2010), 45–56.

[17] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 1996.

[18] Cynthia Whissell. Traditional and emotional stylometric analysis of the songs of Beatles Paul McCartney and John Lennon. *Computers and the Humanities* 30:3 (1996), 257–265.

[19] Susannah Wixey and Rob Sturman. A graph-theoretic approach to efficient voice-leading. arXiv preprint, arXiv:1602.04137, 2016.

Jason I. Brown, Department of Mathematics and Statistics, Dalhousie University, 6316 Coburg Road, Halifax, Nova Scotia B3H 4R2, Canada.
E-mail: jason.brown@dal.ca

Ian C. George, Department of Mathematics and Statistics, Dalhousie University, 6316 Coburg Road, Halifax, Nova Scotia B3H 4R2, Canada. E-mail: icg@dal.ca