Enumerating Convex Sets of Posets

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Background

Notation

All posets will be finite.

For $S \subseteq P$, the *upset* generated by S is

$$V(S) = \{p \in P \ : \ \exists s \in S,\ s \leq p\}$$

Similarly, the *downset* generated by S is

$$\Lambda(S) = \{p \in P \ : \ \exists s \in S,\ p \leq s\}$$

Notation

For $u, v \in P$, the *interval* $[u, v]$ is the set

$$[u, v] = \{w \in P \ : \ u \leq w \leq v\} = V(\{u\}) \cap \Lambda(\{v\})$$

For $S \subseteq P$, $\min(S)$ ($\max(S)$) is the set of minimal (maximal) elements of S.

Motivation: intervals in causal sets

In 2013, Glaser and Surya considered $n = \rho V$-point sprinklings into a finite region of volume V of flat Minkowski space $\mathbb{M}^d$.

The expected quantity of intervals of size $m+2$ in such a sprinkling is

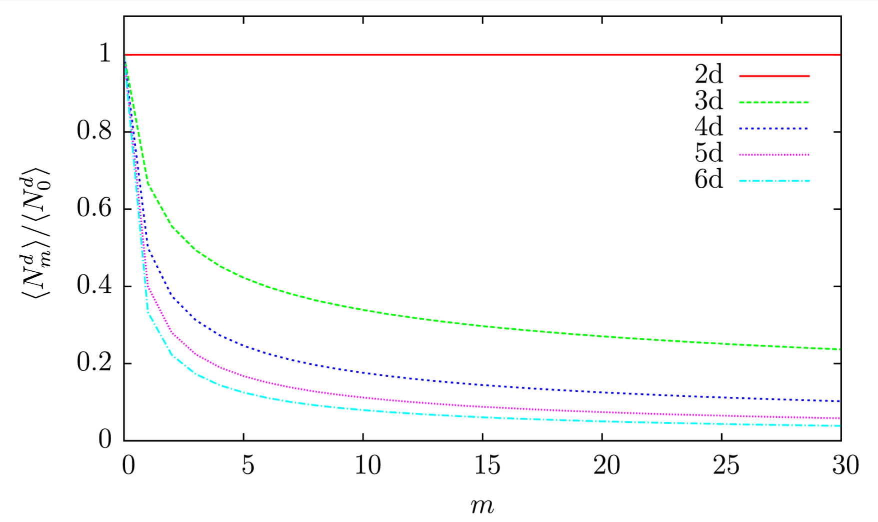
$$\langle N_m^d(\rho, V)\rangle = \text{a hypergeometric function}$$

As n (or density ρ) $\to \infty$, the normalized (N_m/N_0) ratio approaches

$$\mathcal{S}_m^d \equiv \frac{\Gamma(\frac{2}{d}+m)}{\Gamma(\frac{2}{d})\Gamma(m+1)} = \binom{\frac{2}{d}+m-1}{m}$$

$$\implies \sum_{m\geq 0} \mathcal{S}_m^d z^m = \frac{1}{(1-z)^{2/d}}$$

Motivation: intervals in causal sets



L. Glaser and S. Surya. Towards a definition of locality in a manifoldlike causal set. *Physical Review D-Particles, Fields, Gravitation, and Cosmology*, 88(12):124026, 2013.

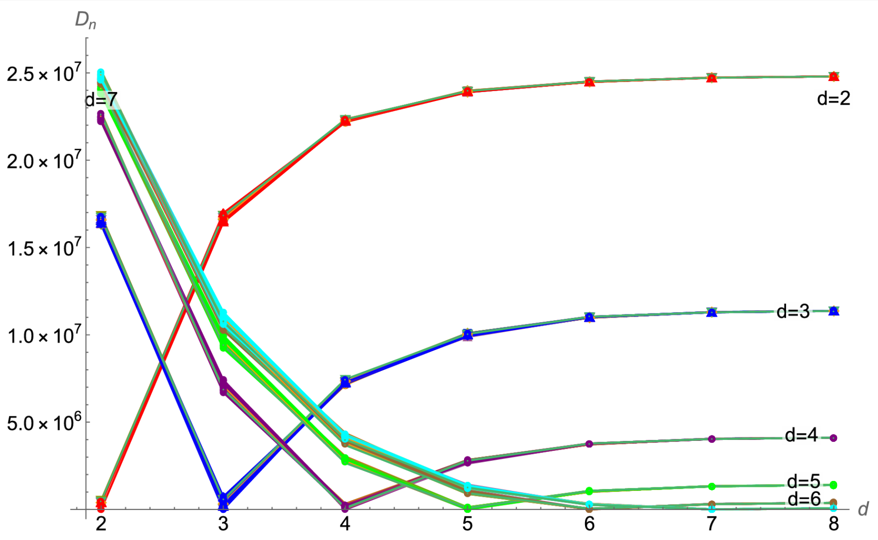
Motivation: intervals in causal sets

Surya (2026) used interval distributions to define a *closeness* function between (finite-volume regions of) spacetimes, or between causal sets:

$$\mathcal{D}_n(\mathcal{M}, \mathcal{M}') = \sum_{m \geq 0} |N_m(\mathcal{M}, n) - N_m(\mathcal{M}', n)|$$

Where $N_m(\mathcal{M}, n)$ is the interval distribution of an n-point sprinkling in $\mathcal{M}$.

Motivation: intervals in causal sets



S. Surya. Closeness function on coarse grained Lorentzian geometries. *Physical Review D*, 113(2):024034, 2026.

Motivation: enumerating intervals

There has been some interest in enumerating intervals in posets that have combinatorial significance:

- m-Tamari lattice (B.-M.–F.–P.–R. 2011, C. 2006)
- Consecutive pattern poset (Elizalde–McNamara 2016)
- Tamari lattice generalization (Préville-Ratelle–Viennot 2017)
- Young's lattice (Azam–Richmond 2021)
- Greedy Tamari poset (Bousquet-Mélou–Chapoton 2023)
- Fibonacci lattice (Baril–Hassler 2024)
- Lattice of q-decreasing words (Baril–Hassler–Kirgizov 2025)

Convex sets

Definition

$X \subseteq P$ is *convex* if for any $u, v \in P$, $u, v \in X$ implies $[u, v] \subseteq X$.

Convex sets are generalizations of intervals in the following sense:

$$X = V(\min(X)) \cap \Lambda(\max(X))$$

Also called *interval-closed sets*, or *generalized intervals*.

A generating polynomial

The polynomial

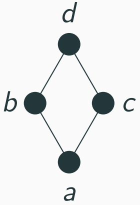
Definition

We define the following generating polynomial for convex sets:

$$\Phi(P; s, t, x, y, z) = \sum_{\substack{X \subseteq P \\ X \text{ convex}}} s^{|\min(X)|} t^{|\max(X)|} x^{|\Lambda(\max(X))|} y^{|V(\min(X))|} z^{|X|}$$

Note: the empty set contributes 1 to the polynomial.

An example



$$\begin{aligned}\Phi(P; s, t, x, y, z) = & 1 + stxyz(x^3y^3z^3 + sx^3y^2z^2 + tx^2y^3z^2 \\ & + stx^2y^2z + 2x^3yz + 2xy^3z + x^3 + y^3 + 2xy)\end{aligned}$$

Information it encodes

Poset information/property	How to obtain it
Number of elements	$[stz]\Phi(P; s, t, 1, 1, z)$, or maximum n such that $[z^n]\Phi(P; s, t, x, y, z) \neq 0$
Number of cover relations	$[stz^2]\Phi(P; s, t, 1, 1, z)$
Number of relations	$[st]\Phi(P; s, t, 1, 1, 1)$
Number of minimal elements	m such that $[s^m y^{\lvert P\rvert}]\Phi(P; s, t, x, y, z) \neq 0$
Number of maximal elements	m such that $[t^m x^{\lvert P\rvert}]\Phi(P; s, t, x, y, z) \neq 0$
Number of antichains of size k	$[s^k t^k z^k]\Phi(P; s, t, 1, 1, z)$
Width	Maximum k such that $[s^k t^k z^k]\Phi(P; s, t, x, y, z) \neq 0$

Information it encodes

Generating polynomial for sizes of...	How to obtain it
Convex sets	$\Phi(P; 1, 1, 1, 1, z)$
Upsets, downsets, convex sets	$\Phi(P; 1, 1, x, y, z)$
Upsets	$\Phi(P; 1, 1, 1, yw, 1/w)\|_{w=0}$
Downsets	$\Phi(P; 1, 1, xw, 1, 1/w)\|_{w=0}$
Principal upsets	$[s]\Phi(P; s, 1, 1, yw, 1/w)\|_{w=0}$
Principal downsets	$[t]\Phi(P; 1, t, 1, yw, 1/w)\|_{w=0}$
Principal upsets and downsets	$[z]\Phi(P; 1, 1, x, y, z)$
Intervals	$[st]\Phi(P; s, t, 1, 1, z)$
Principal upsets, downsets, intervals	$[st]\Phi(P; s, t, x, y, z)$
Antichains	$\Phi(P; 1/w, 1, 1, 1, zw)\|_{w=0}$

Why this form?

- Include the upsets/downsets to get past/future invariant
- Antichain sizes for spatial information

Results from literature

Theorem (Barnette–Nichols–Richmond 2019)

$$\Phi([n]\times[m];\mathbf{1}) = \frac{1}{n+1}\sum_{r=0}^{n-1}\sum_{q=0}^{r}(n-r)\binom{r}{q}\binom{n+q+1}{q+1}\binom{m+r+1}{r+q+2}$$

Theorem (E.–L.–L.–M.–S.–W. 2025)

$$\sum_{m,n\geq 0}\Phi([n]\times[m];\mathbf{1})u^m v^n = \frac{2}{1-u-v+2uv+\sqrt{(1-u-v)^2-4uv}}$$

Some formulas

Disjoint union of posets:

Proposition (G.–Yeats 2025)

$\Phi(P+Q;s,t,x,y,z)=\Phi(P;s,t,x,y,z)\Phi(Q;s,t,x,y,z)$

Ordinal sum of posets:

Proposition (G.–Yeats 2025)

$$\begin{aligned}\Phi(P\oplus Q;s,t,x,y,z) &= 1+y^{|Q|}(\Phi(P;s,t,x,y,z)-1)\\ &\quad + x^{|P|}(\Phi(Q;s,t,x,y,z)-1)\\ &\quad + x^{|P|}y^{|Q|}\Phi(P;s,1,1,yw,z/w)|_{w=0}\\ &\qquad\qquad \cdot \Phi(Q;1,t,xw,1,z/w)|_{w=0}\end{aligned}$$

Some formulas

Example

For P an antichain of n elements, $\Phi(P; s, t, x, y, z) = (1 + stxyz)^n$.

Example

If $P = [n]$ is a chain consisting of n elements, then

$$\begin{aligned}
\Phi(P; s, t, x, y, z) &= 1 + st \sum_{k=1}^{n} z^k \sum_{i=0}^{n-k} x^{k+i} y^{n-i} \\
&= 1 + st \sum_{k=1}^{n} z^k \frac{x^{n+1} y^k - x^k y^{n+1}}{x - y} \\
&= 1 + \frac{st}{x-y} \left(x^{n+1} \frac{1 - (yz)^{n+1}}{1 - yz} - y^{n+1} \frac{1 - (xz)^{n+1}}{1 - xz} \right)
\end{aligned}$$

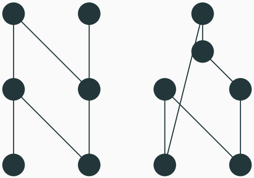
Distinguishing

Distinguishing posets

Question

Q: Does Φ distinguish all posets?

A: No!



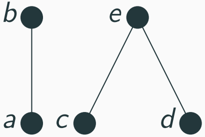
However...

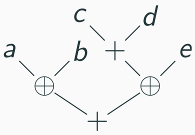
Series-Parallel posets

Definition

A Series-Parallel (SP) poset is a poset that is built recursively via $+$ and $\oplus$, starting with single elements. Equivalently, it is an N-free poset.

The recursive tree structure (*composition tree*) of $+$ and $\oplus$ operations used to make a SP poset determines it completely.





Series-Parallel Theorem

Theorem (G.–Yeats 2025)

If P is an SP poset, then P can be uniquely reconstructed from $\Phi(P; s, t, x, y, z)$.

Corollary (G.–Yeats 2025)

Given SP posets P and Q, if $\Phi(P; s, t, x, y, z) = \Phi(Q; s, t, x, y, z)$, then $P \cong Q$.

Theorem proof outline

- Show that the Φs of ordinal summands can be obtained from Φ
- Show that Φ of an ordinal sum does not factor* into smaller polynomials
- Inductively determine the composition tree of a SP poset from Φ

Theorem proof

Lemma 1

Let $P = P_1 \oplus \cdots \oplus P_r$ be an ordinal sum of posets where each P_i cannot be written as an ordinal sum of strictly smaller posets. Then we can recover $|P_1|, \ldots, |P_r|$ from $\Phi(P; s, t, x, y, z)$.

Proof sketch: Algorithm which identifies the terms of Φ corresponding to cover relations between some P_ℓ and $P_{\ell+1}$ and finds each $|P_\ell|$ from the x-exponents.

Theorem proof

Lemma 2

Let $P_1, \ldots, P_r$ and P be as before. Then for each $1 \leq \ell \leq r$, we can recover $\Phi(P_\ell; s, t, x, y, z)$ from $\Phi(P; s, t, x, y, z)$.

Proof sketch: To get a (non-constant) term of $\Phi(P_\ell; s, t, x, y, z)$, take a term $cs^\alpha t^\beta x^\gamma y^\delta z^\varepsilon$ of $\Phi(P; s, t, x, y, z)$ such that $\gamma > |P_1| + \cdots + |P_{\ell-1}|$ and $\delta > |P_{\ell+1}| + \cdots + |P_r|$. Replace γ with $\gamma - |P_1| - \cdots - |P_{\ell-1}|$ and δ with $\delta - |P_{\ell+1}| - \cdots - |P_r|$.

Theorem proof

Lemma 3

Let $P = P_1 \oplus P_2$ for any non-empty posets P_1, P_2. Then $\Phi(P; s, t, x, y, z)$ does not factor into two non-constant polynomials each with constant coefficient 1.

Proof sketch: For contradiction, suppose $\Phi(P; s, t, x, y, z) = (1 + F)(1 + G)$. We can find a term $T = T_F T_G$ which belongs to FG, such that T_F and T_G are terms corresponding to antichains. No matter where these antichains lie (i.e. in P_1 or P_2), we find a contradiction with the exponents being too large.

Theorem proof

Putting it all together using induction on $|P|$:

If $|P| = 1$, $P = \bullet$.

Otherwise, suppose the result holds for SP posets with fewer than $|P|$ elements. Minimally factor $\Phi(P; s, t, x, y, z) = F_1 F_2 \cdots F_r$, scaled so all factors have constant coefficient 1. From Lemma 3 it follows that P has exactly r components, i.e. $P = P_1 + \cdots + P_r$. From the disjoint union Proposition, we have $F_i = \Phi(P_i; s, t, x, y, z)$ (after some labelling of the $P_1, \ldots, P_r$). For each i, P_i is an ordinal sum. So $P_i = Q_{i,1} \oplus \cdots \oplus Q_{i,\ell}$, where each $Q_{i,j}$, $1 \leq j \leq \ell$, is not an ordinal sum of smaller, non-empty posets. By Lemma 2, we can recover $\Phi(Q_{i,j}; s, t, x, y, z)$ for each j from $\Phi(P_i; s, t, x, y, z)$. By induction, we can uniquely determine each $Q_{i,j}$, and hence P.

Causal Set Theory

In Causal Set Theory

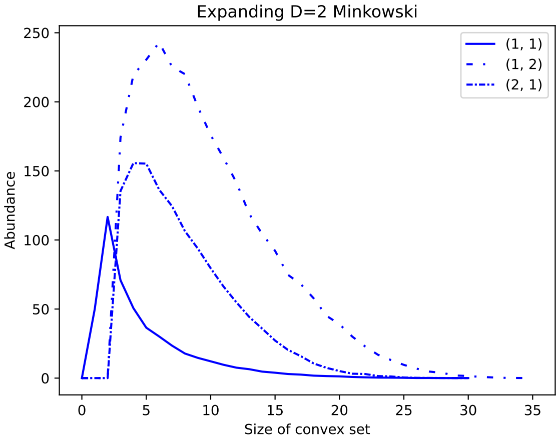
The distribution of interval sizes contains important physical information about sprinkled causal sets.

However, there are some shortcomings:

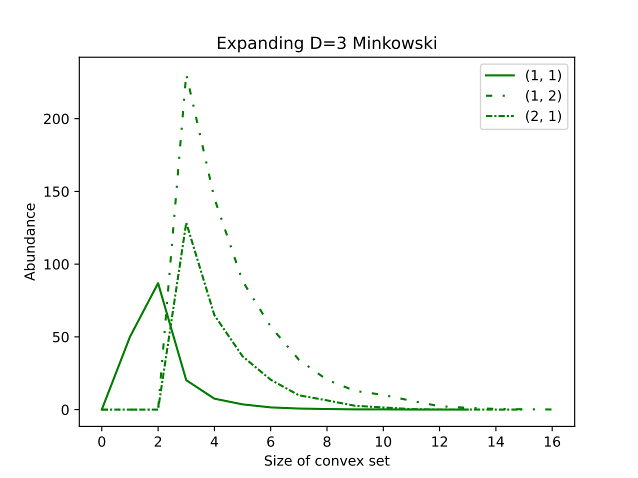
- Time reversal, or expanding/contracting spacetimes
- Spacial information
- Slow convergence

Can convex sets offer any more insight?

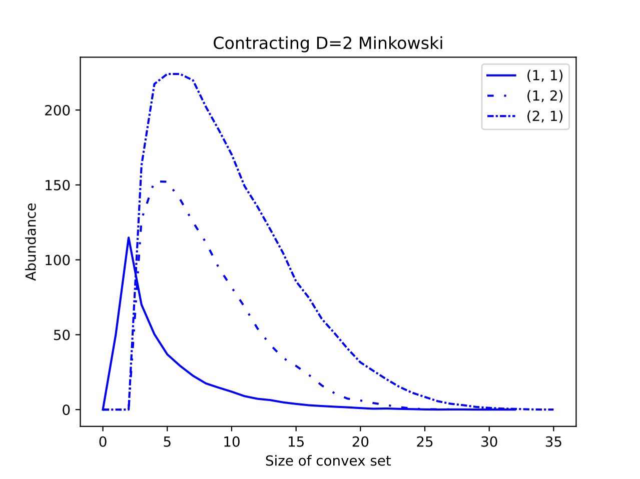
Detecting expansion



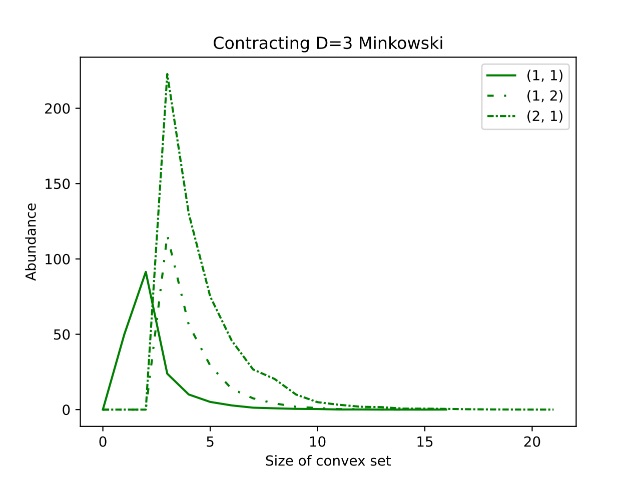
Detecting expansion



Detecting contraction



Detecting contraction



Future work

On the combinatorics side...

- More exact formulas for Φ ($[n] \times [m]$, etc.)
- Convex partitions?

On the CST side...

- More/larger simulations
- Define a *closeness* function with convex set distributions

Thank you!

arXiv:2511.14924