Some Poset Invariants for Causal Sets

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Outline

- ◊ Background
- ◊ Convex sets
- ◊ Jordan partitions
- ◊ Future work

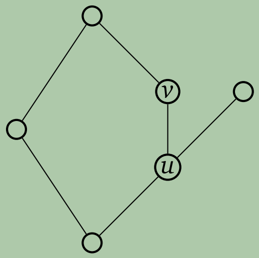
Background

Posets

A partial order is a set P with a binary relation $\leq$ that is $(u, v, w \in P)$...

- ◊ Reflexive: $u \leq u$
- ◊ Anti-symmetric: $u \leq v$ and $v \leq u \implies u = v$
- ◊ Transitive: $u \leq v$ and $v \leq w$ imply $u \leq w$

Posets are drawn graphically by their Hasse diagram, with u below v for a relation $u \leq v$.



Poset terminology

For us, P will be finite.

A relation $u < v$ where there is no w such that $u < w < v$ is a *cover relation/link*.

P is a *chain* when every pair of distinct elements is comparable, and an *antichain* when none of them are.

The *height* $ht(P)$ of P is the maximum length of a chain in P, and the *width* is the maximum size of an antichain in P.

An *interval* is a subset $[u, v] = \{w \in P : u \leq w \leq v\}$.

For $S \subseteq P$, the *downset* generated by S is $\Lambda(S) = \{w \in P : w \leq s \text{ for some } s \in S\}$. The *upset* $V(S)$ generated by S is defined similarly.

For $S \subseteq P$, $\min(S)$ and $\max(S)$ denote the sets of minimal and maximal elements of S, respectively.

The causal set hypothesis

Spacetime is a locally-finite poset (*causal set*) [Bombelli+ 1987].

Lorentzian manifold	Locally-finite poset
Causal relation	Partial order
Volume	Number of elements

Order + Number $\sim$ Geometry

Convex sets

Motivation: intervals in causal sets

In 2013, Glaser and Surya [Glaser+ 2013] considered $n = \rho V$-point sprinklings into a causal diamond of volume V in flat Minkowski space $\mathbb{M}^d$.

The expected quantity of intervals of size $m+2$ in such a sprinkling is

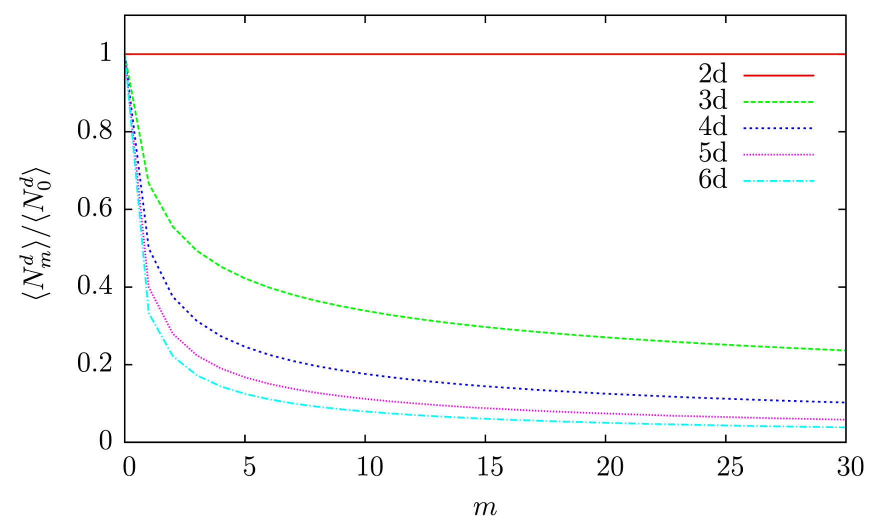
$$\langle N_m^d(\rho, V)\rangle = \text{a hypergeometric function}$$

As n (or density ρ) $\to \infty$, the normalized (N_m/N_0) ratio approaches

$$\mathscr{S}_m^d \equiv \frac{\Gamma(\frac{2}{d}+m)}{\Gamma(\frac{2}{d})\Gamma(m+1)} = \binom{\frac{2}{d}+m-1}{m}$$

$$\implies \sum_{m\geq 0} \mathscr{S}_m^d z^m = \frac{1}{(1-z)^{2/d}}$$

Motivation: intervals in causal sets



L. Glaser and S. Surya. Towards a definition of locality in a manifoldlike causal set. *Physical Review D-Particles, Fields, Gravitation, and Cosmology*, 88(12):124026, 2013.

A generating polynomial for convex sets

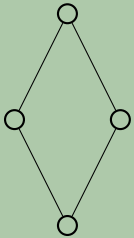
$X \subseteq P$ is *convex* if $u, v \in X$ implies $[u, v] \subseteq X$. Convex sets generalize intervals in a natural way.

We define the following generating polynomial for convex sets:

$$\Phi(P; s, t, x, y, z) = \sum_{\substack{X \subseteq P \\ X \text{ convex}}} s^{|\min(X)|} t^{|\max(X)|} x^{|\Lambda(\max(X))|} y^{|V(\min(X))|} z^{|X|}$$

This polynomial generalizes the generating polynomial for intervals, and includes the *pf*-invariant suggested by Sorkin [Sorkin 2024].

An example



$$\Phi(P; s, t, x, y, z) = 1 + stxyz(x^3y^3z^3 + sx^3y^2z^2 + tx^2y^3z^2 + stx^2y^2z + 2x^3yz + 2xy^3z + x^3 + y^3 + 2xy)$$

Information it encodes

Poset information/property	How to obtain it
Number of elements	$[stz]\Phi(P; s, t, 1, 1, z)$, or maximum n such that $[z^n]\Phi(P; s, t, x, y, z) \neq 0$
Number of cover relations	$[stz^2]\Phi(P; s, t, 1, 1, z)$
Number of relations	$[st]\Phi(P; s, t, 1, 1, 1)$
Number of minimal elements	m such that $[s^m y^{\lvert P\rvert}]\Phi(P; s, t, x, y, z) \neq 0$
Number of maximal elements	m such that $[t^m x^{\lvert P\rvert}]\Phi(P; s, t, x, y, z) \neq 0$
Number of antichains of size k	$[s^k t^k z^k]\Phi(P; s, t, 1, 1, z)$
Width	Maximum k such that $[s^k t^k z^k]\Phi(P; s, t, x, y, z) \neq 0$

Information it encodes

Generating polynomial for sizes of...	How to obtain it
Convex sets	$\Phi(P;1,1,1,1,z)$
Upsets, downsets, convex sets	$\Phi(P;1,1,x,y,z)$
Upsets	$\Phi(P;1,1,1,yw,1/w)\rvert_{w=0}$
Downsets	$\Phi(P;1,1,xw,1,1/w)\rvert_{w=0}$
Principal upsets	$[s]\Phi(P;s,1,1,yw,1/w)\rvert_{w=0}$
Principal downsets	$[t]\Phi(P;1,t,1,yw,1/w)\rvert_{w=0}$
Principal upsets and downsets	$[z]\Phi(P;1,1,x,y,z)$
Intervals	$[st]\Phi(P;s,t,1,1,z)$
Principal upsets, downsets, intervals	$[st]\Phi(P;s,t,x,y,z)$
Antichains	$\Phi(P;1/w,1,1,1,zw)\rvert_{w=0}$

Properties of the polynomial

Disjoint union:

$$\Phi(P+Q;s,t,x,y,z) = \Phi(P;s,t,x,y,z)\Phi(Q;s,t,x,y,z)$$

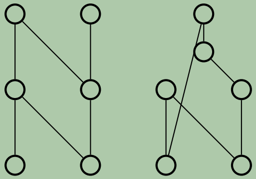
Ordinal sum:

$$\begin{aligned}
&\Phi(P \oplus Q;s,t,x,y,z) \\
&= y^{|Q|}(\Phi(P;s,t,x,y,z)-1) + x^{|P|}(\Phi(Q;s,t,x,y,z)-1) \\
&\quad + x^{|P|}y^{|Q|}(\Phi(P;s,1,1,yw,z/w)|_{w=0}-1)(\Phi(Q;1,t,xw,1,z/w)|_{w=0}-1)+1
\end{aligned}$$

Cartesian product: no formula (provably).

Properties of the polynomial

It does not distinguish all posets:



However, we know how Φ behaves with union and ordinal sum. Starting with single elements, these operations form series-parallel (SP) posets.

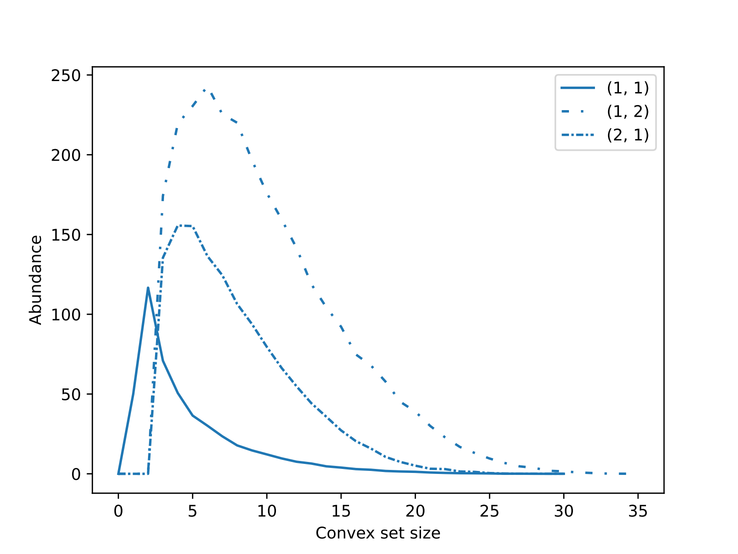
Theorem [George + 2025]

Given SP posets P and Q, if $\Phi(P; s, t, x, y, z) = \Phi(Q; s, t, x, y, z)$, then $P \cong Q$.

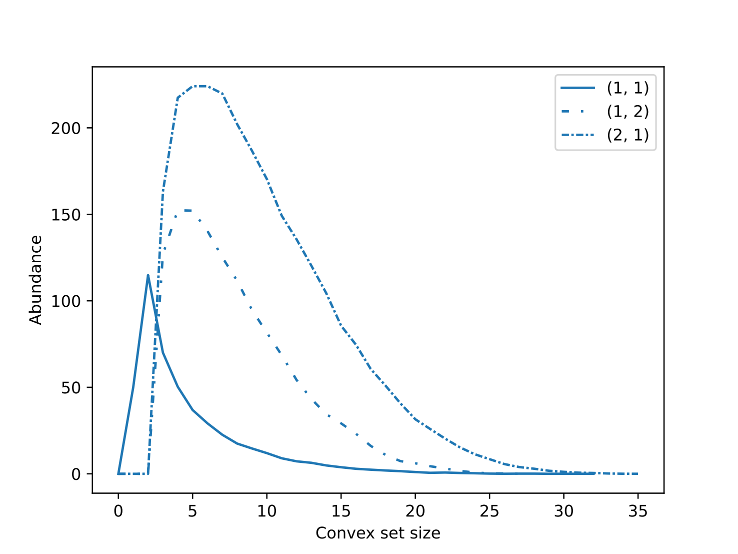
For causal sets

- ◊ Parameters s, t allow distinguishing of temporal direction
  - Convex X with $(|\min(X)|, |\max(X)|) = (a, b)$ versus (b, a)
- ◊ Parameters x, y position convex sets temporally
- ◊ Convex sets access more space than intervals

Data: temporal symmetry (2D expanding)



Data: temporal symmetry (2D contracting)

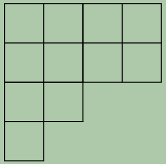


Jordan partitions

Integer partitions

A *partition* of $N \in \mathbb{N}_{>0}$ is a sequence $\lambda = (\lambda_i)_i$ such that $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ and $\sum_i \lambda_i = N$.

Often represented with *Young diagrams*:



$$\lambda = 4, 4, 2, 1 \vdash 11$$

Integer partitions

- ◊ Representation theory of finite symmetric groups
  - – Irreducible representations of S_N exactly correspond to partitions of N
- ◊ SYT/SSYT, Schur polynomials, algebraic combinatorics at large
- ◊ Statistical mechanics
- ◊ Random matrices

Jordan partitions

Recall that if M is an $N \times N$ matrix, we can determine the *Jordan form* $P^{-1}MP = J$:

$$J = \begin{pmatrix} J_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & J_k \end{pmatrix}, \quad J_i = \begin{pmatrix} \mu_i & 1 & 0 & \cdots & 0 \\ 0 & \mu_i & 1 & \cdots & 0 \\ 0 & 0 & \mu_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_i \end{pmatrix}$$

where μ_i are the eigenvalues of M.

Thus we get the *Jordan partition* $J(M) : |J_1| \geq |J_2| \geq \cdots \geq |J_k| > 0$.

Jordan partitions for posets

To P we associate a nilpotent matrix C:

$$C_{u,v} = \begin{cases} x_{uv} & \text{if } u < v \\ 0 & \text{otherwise} \end{cases}$$

When the x_{uv} are independent transcendentals (or indeterminates), $C = C^{gen}$ is *generic causal*. When $x_{uv} = 1$ for each $u < v$, $C = C^{caus}$ is *causal*.

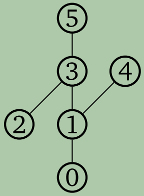
$\lambda^{gen} = J(C^{gen})$, and $\lambda^{caus} = J(C^{caus})$.

Jordan partitions for posets

Let p_i be the largest cardinality of i disjoint chains of P ($p_0 = 0$).

It is known [Saks 1980; Gansner 1981] that $\lambda_i^{gen} = p_i - p_{i-1}$, for $i \geq 1$. This is also called the *Greene/Greene-Kleitman* shape of P.

Note that a realizer of p_i does not necessarily contain a realizer of p_{i-1}:



$$\lambda^{gen} = 4, 2$$
$$p_1 : \{0, 1, 3, 5\}$$
$$p_2 : \{0, 1, 4\}, \{2, 3, 5\}$$

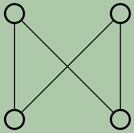
Jordan partitions for posets

A similar precise statement for λ^{caus} is not known. We do know [Saks 1980; Gansner 1981] that

$$\lambda^{caus} \leq_D \lambda^{gen}$$

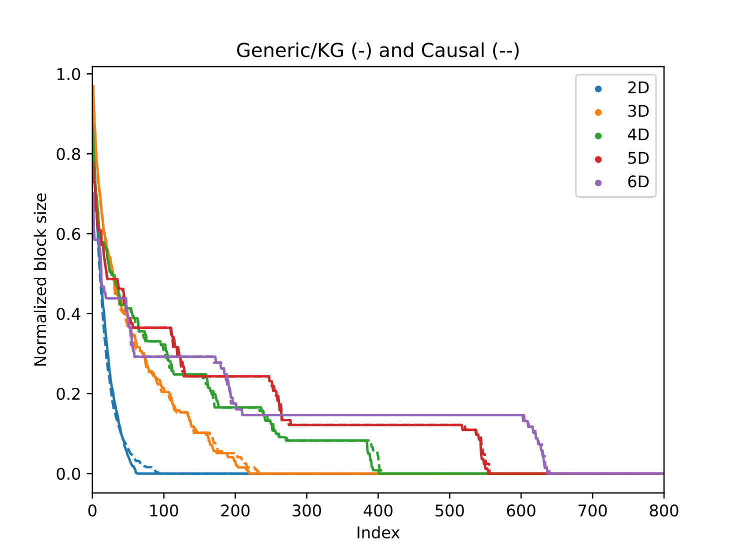
where $\leq_D$ is the *dominance/majorization* order on integer partitions: $\sum_1^k \lambda_i^{caus} \leq \sum_1^k \lambda_i^{gen}$ for all k.

(Work in progress) the discrepancy seems to be caused by the presence of bowties (with the edges subdivided into paths, in general):



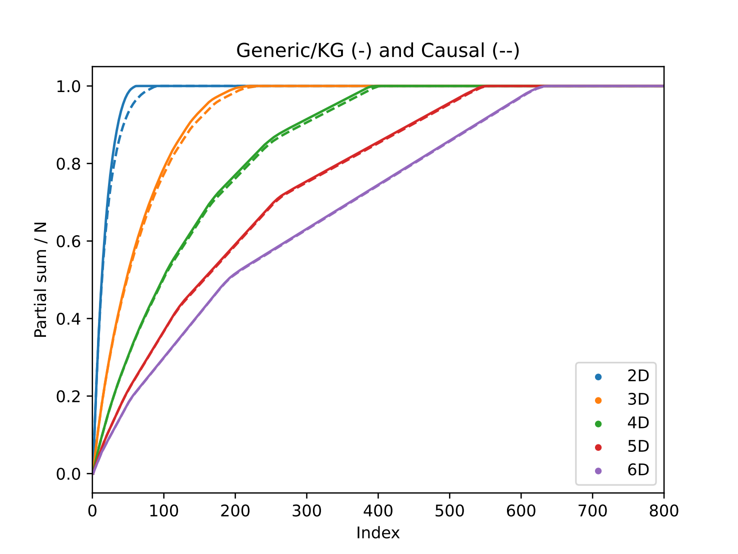
$$\lambda^{gen} = 2, 2$$
$$\lambda^{caus} = 2, 1, 1$$

Data: sprinklings



y-scale: $\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}Nd\cdot d!!\right)^{1/d}$

Data: partial sums



y-scale: $\frac{1}{N}$

The 2D case

A sprinkling of N points into a 2D Minkowski diamond generates a permutation $\sigma \in S_N$ uniformly at random (project onto the boundary).

The sprinkled poset is isomorphic to the poset P_σ induced by σ:

- ◊ $i <_P j$ if $i < j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$

Greene [Greene 1974] showed that $\lambda^{gen}(P_\sigma)$ is the shape of the Young diagram obtained by putting σ through the *RS/RSK correspondence*.

Furthermore, this is the expected partition when sampling with respect to the *Plancherel measure* on S_N.

- ◊ $\mathbb{P}_{Planch}(\lambda \vdash N) = \frac{(f^\lambda)^2}{N!}$
- ◊ $f^\lambda = \dim(\lambda) = \frac{N!}{\prod(\textbf{hook length})} = \#SYT$ of shape λ

The 2D case

Logan+Shepp [Logan+ 1977] and Vershik+Kerov [Vershik+ 1985; Vershik+ 1977] determined the expected shape as $N \to \infty$ after scaling. The parametrized cartesian coordinates of this shape are [Romik 2015]:

$$x(\theta) = \left(\frac{2\theta}{\pi} + 1\right)\sin\theta + \frac{2}{\pi}\cos\theta$$

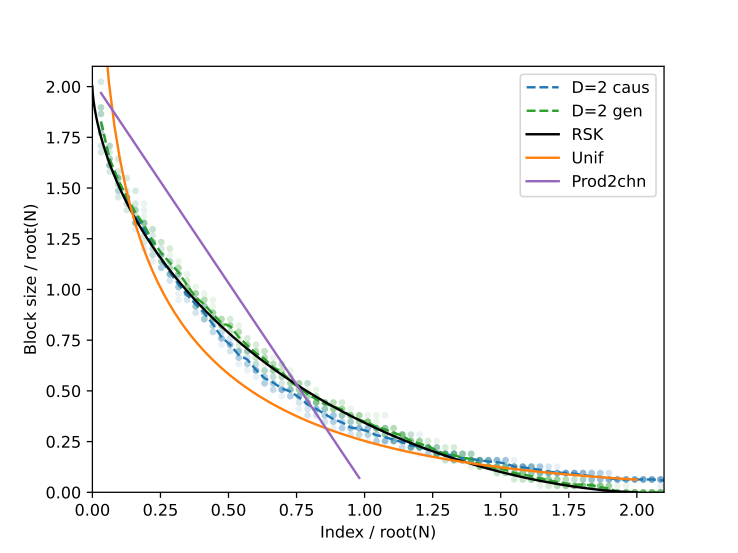
$$y(\theta) = \left(\frac{2\theta}{\pi} - 1\right)\sin\theta + \frac{2}{\pi}\cos\theta$$

for $\theta \in [-\pi/2, \pi/2]$.

In contrast, the expected shape of a partition $\lambda \vdash N$ chosen uniformly, as $N \to \infty$ (and scaled similarly) [Vershik 1996] is:

$$\exp\left(-\frac{\pi}{\sqrt{6}}x\right) + \exp\left(-\frac{\pi}{\sqrt{6}}y\right) = 1$$

Data: 2D



x, y-scale: $\frac{1}{\sqrt{N}}$

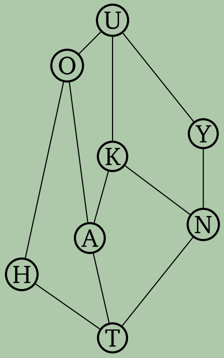
Future work

Convex sets:

- $\lozenge$ More distinguishing results
- $\lozenge$ Probe deeper into the spacetime topology and varying dimensions (with Surya)
- $\lozenge$ Analytical calculations (with Surya)?
- $\lozenge$ Construct a distance function à la Surya

Jordan partitions:

- $\lozenge$ Investigate different spacetime topologies, e.g. de Sitter
- $\lozenge$ Determine the expected λ^{gen} for sprinklings in $d > 2$ Minkowski diamonds, and the $N \to \infty$ limit
  - – Determine the probability distribution $\mathbb{P}_d(\lambda \vdash N)$
- $\lozenge$ Pin down the statement/interpretation for λ^{caus}
- $\lozenge$ What about λ^{link}?
- $\lozenge$ Eason Li: distribution of λs for small subposets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