Enumerating Convex Sets in Posets

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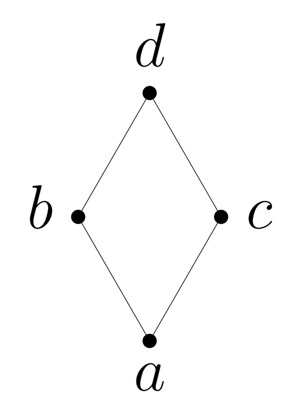
Definitions

Let P be a finite poset. For $S \subseteq P$, $V(S)$ and $\Lambda(S)$ denote the upset and downset generated by S, respectively. $\min(S)$ and $\max(S)$ are the sets of minimal and maximal elements of S, respectively. For $u, v \in P$, $[u, v]$ is the interval from u to v. $X \subseteq P$ is *convex* if $u, v \in X$ imply $[u, v] \subseteq X$.

The polynomial

$$\Phi(P;\mathbf{x}) = \sum_{\substack{X \subseteq P \\ X \text{ convex}}} s^{|\min(X)|} t^{|\max(X)|} x^{|\Lambda(\max(X))|} y^{|V(\min(X))|} z^{|X|}$$

Example



$$\begin{aligned}\Phi(P;\mathbf{x}) = 1 &+ stxyz(x^3y^3z^3 + sx^3y^2z^2 \\ &+ tx^2y^3z^2 + stx^2y^2z \\ &+ 2x^3yz + 2xy^3z \\ &+ x^3 + y^3 + 2xy)\end{aligned}$$

Specializations

Number of elements	$[stz]\Phi(P; s, t, 1, 1, z)$, or maximum n s.t. $[z^n]\Phi(P; s, t, x, y, z) \neq 0$
... cover relations	$[stz^2]\Phi(P; s, t, 1, 1, z)$
... relations	$[st]\Phi(P; s, t, 1, 1, 1)$
... minimal elements	m s.t. $[s^m y^{\lvert P\rvert}]\Phi(P; s, t, x, y, z) \neq 0$
... antichains of size k	$[s^k t^k z^k]\Phi(P; s, t, 1, 1, z)$
Upsets, downsets, convex sets	$\Phi(P; 1, 1, x, y, z)$
Upsets	$\Phi(P; 1, 1, 1, yw, 1/w)\rvert_{w=0}$
Intervals	$[st]\Phi(P; s, t, 1, 1, z)$
Antichains	$\Phi(P; 1/w, 1, 1, 1, zw)\rvert_{w=0}$

Previous work

Previous work focuses on enumerating intervals in certain families of posets (see, for example, [1, 2, 3, 4]). For convex sets, only the posets $[n] \times [m]$ have received attention [5, 6].

Series-Parallel

A poset is series-parallel (SP) if it contains no subposet isomorphic to the N-poset. Equivalently, it is constructed recursively with the $+$ and $\oplus$ operations, starting with single elements. An SP poset is completely determined by the *composition tree* of $+$ and $\oplus$ operations.

Union and Ordinal sum

$$\Phi(P + Q;\mathbf{x}) = \Phi(P;\mathbf{x})\Phi(Q;\mathbf{x})$$

$$\begin{aligned}\Phi(P \oplus Q;\mathbf{x}) = 1 &+ y^{|Q|}(\Phi(P;\mathbf{x}) - 1) \\ &+ x^{|P|}(\Phi(Q;\mathbf{x}) - 1) \\ &+ x^{|P|}y^{|Q|}\Phi(P; s, 1, 1, yw, z/w)|_{w=0} \\ &\cdot \Phi(Q; 1, t, xw, 1, z/w)|_{w=0}\end{aligned}$$

Theorem (G.–Y., 2025)

For P, Q series-parallel, if $\Phi(P;\mathbf{x}) = \Phi(Q;\mathbf{x})$ then $P \cong Q$.

Proof sketch.

- Show that the Φs of ordinal summands can be obtained from Φ
- Show that Φ of an ordinal sum does not factor* into smaller polynomials
- Inductively determine the composition tree of a SP poset from Φ

$\square$

Causal Set Theory (CST)

Causal Set Theory (CST) is a theory of quantum gravity in which spacetime is taken to be a locally-finite poset, or a *causal set*: the causal relation determined by the metric induces a partial order on the elements of the spacetime. This framework allows for investigating physically relevant properties by combinatorial means. For example, the distribution of interval sizes captures dimensional information of the spacetime [7]. These distributions can also be used to define a *closeness* function on causal sets that identifies global topology [8]. A future avenue of work will be to extend these previous studies to include convex sets, not just intervals. Early results indicate that distributions of convex set sizes identify the expanding/contracting nature of a Minkowski spacetime.

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